

Uniqueness of reconstruction and an inversion procedure for thermoacoustic and photoacoustic tomography

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Abstract

The paper contains a simple approach to reconstruction in Thermoacoustic and Photoacoustic Tomography. The technique works for any geometry of point detectors placement and for variable sound speed satisfying a non-trapping condition. A uniqueness of reconstruction result is also obtained.

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1 Introduction

In the past decade, one witnessed a surge in newly developing medical imaging modalities. Their designers pursue the lofty goals of increasing the image resolution, contrast, and safety, while reducing the costs. A new approach evident in some of these developments is based upon combining different physical types of signals in one procedure, with the hope of reducing the deficiencies of each individual one, and at the same time taking advantage of the strengths of each. The most well developed example is the **Thermoacoustic Tomography (TAT)** (also abbreviated as TCT) [19]. Let us

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provide a brief description of the TAT procedure (also see [11, 18, 19, 22, 35] and [40]-[45]).

A very short radiofrequency (RF) pulse is sent through a biological object. The whole object is assumed to be uniformly irradiated. Some RF energy

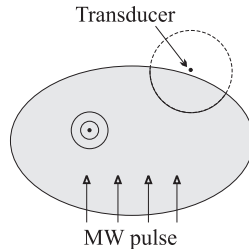


Figure 1: The TAT procedure.

is absorbed at each location inside the object. It is known (e.g., [19, 41]) that cancerous cells absorb several times more energy in the RF range than the healthy ones. Thus, if we knew the distribution of the absorption of RF energy function $f(x)$, this high contrast could provide an easy identification of cancerous locations. However, RF waves with sufficient penetration depth are too long to lead to a good resolution, if used for imaging. Thus, a different mechanism is used for imaging $f(x)$. Namely, energy absorption causes thermoelastic expansion of the tissue, which in turn leads to propagation of a pressure wave $p(x, t)$. This signal is measured by transducers distributed according to some specifically chosen geometry around the object. Ultrasound imaging has rather low contrast, but in the TAT procedure, this is not a problem, since the contrast comes from the electromagnetic absorption, and ultrasound is only responsible for high resolution, which it does have. Thus, recovery of $f(x)$ from the measured characteristics of the pressure $p(x, t)$, is the main goal of TAT.

The smallness of the ultrasound contrast is in fact a good thing for TAT. For instance, it has been mostly assumed that the sound speed is constant. This is the main assumption under which most known mathematical developments have taken place. In many cases, though, it is important to take variable sound speed into account, and we will do so in this paper.

The photoacoustic tomography (PAT) is almost identical to TAT. The difference is only in how the thermoacoustic signal is generated. In PAT, a laser pulse is used instead of an RF one to initiate the signal [40]. The rest

of the procedure stays the same. Since from the mathematical point of view, there is no difference between TAT and PAT, we will be mentioning TAT only.

Close attention has been paid to developing mathematical methods required for TAT (almost solely under the constant sound speed assumption). The mathematical problems of TAT happen to be very interesting and involving various areas of analysis, in particular PDEs, integral geometry, microlocal analysis, and spectral theory. Many of them, however, also are rather complex (see, e.e. the survey [22] and references therein)¹. After a substantial effort, major breakthroughs have occurred in the last couple of years in all relevant issues: uniqueness of reconstruction, inversion formulas and algorithms, range conditions, incomplete data problems, and stability. The hardest to come about were explicit inversion formulas, as well as some uniqueness of reconstruction results (albeit, for practical geometries, these can be considered to be resolved [3, 22]). For quite a while, only series expansions had been available for the case of transducers placed along a sphere S surrounding the object [31, 32]. No formulas were available for non-spherical surfaces S . The first explicit inversion formulas were obtained in [11] for the spherical geometry in odd dimensions and then extended to even dimensions in [10]. A different (non-equivalent) set of formulas was developed for any dimension in [24] (see also such a formula in 3D in [42]). All these formulas are valid for the spherical geometry only. Another drawback of the inversions of [10, 11, 24, 42] is that if the source $f(x)$ extends beyond the observation surface, the reconstruction even of its part that is inside, becomes incorrect (see discussion of this phenomenon in [22]). A series expansion formula that involves the spectrum and eigenfunctions of the Dirichlet Laplacian was obtained in [25]. Unlike the formulas of [10, 11, 24, 42], it applies to any geometry of measurement and also does not assume that the signal comes from the interior of the measurement surface only. It was also shown in [25] that the eigenfunction expansions method can be implemented to provide fast and accurate reconstructions.

One needs to mention that in the case of the constant sound speed, the reconstruction task can be described as an integral geometry problem dealing with a spherical mean operator [2, 22] (see also [9, 13, 15, 16, 21, 23, 28, 29, 34] and references therein for integral geometric methods). This relation with integral geometry all but disappears for non-constant speed.

¹We will describe the mathematical set-up of TAT in the next section.

No explicit inversion formulas are known for non-spherical surfaces or for the case of variable sound speed. There are also some mysteries surrounding existing formulas. E.g., why do explicit formulas reconstruct incorrectly the function inside the observation surface, if the function has a part outside the surface? Why do inversion formulas exist, even so the general machinery of the so called κ -operator [12, 13, 14] suggests that one should not expect such formulas to appear (since one deals with a so called “inadmissible complex”)? Some of these issues are addressed in [22].

The aim of this article is to develop a very simple general operator theory inversion formula, which applies to arbitrary geometry of the observation surface (i.e., the surface where transducers are placed), variable sound speed under a non- trapping condition, and functions not necessarily supported inside the observation surface. Since the formula involves functions of the Dirichlet Laplacian inside S , it is not easy to apply. We show how it works in some special cases. In particular, it leads to eigenfunction expansion methods that apply in a wider generality (e.g., in non- homogeneous media) than those of [25].

One can mention that mathematical problems similar to the ones of TAT arise in sonar and radar research (e.g., [27, 30]).

The structure of the article is as follows. The next Section 2 sets up the mathematical formulation of the problem. Then, in Section 3, we obtain the general inversion formulas. For the clarity sake, they are first derived under the assumption of a constant sound speed and for a domain of an arbitrary shape in an odd dimensional space \mathbb{R}^n , so the Huygens’ principle holds. Later on, use of the Huygens’ principle is replaced by local energy decay estimates [38, 39]. It is also assumed that functions involved are smooth, a restriction that can easily be lifted after the final formulas are derived. Then an abstract result concerning hyperbolic equations in Hilbert spaces is derived that generalizes the one presented in this simple example. This general formula then is applied to the case of reconstruction in TAT without assumption of constant sound speed. What is relevant, is not the uniformity of the background medium, but rather a non-trapping condition. Such a condition is absolutely natural, since without it one cannot expect any inverse problems of the kind we study to be reasonably solvable. We also establish a uniqueness of reconstruction result. It shows that the data collected is sufficient for recovery of a (compactly supported) function, even if its support reaches outside the observation surface (albeit the inversion procedure recovers only its interior part). The next Section 4 contains additional remarks and discussions. This

is followed by an Acknowledgements section.

2 Mathematical model of TAT

Let the function $f(x)$ to be reconstructed be compactly supported in \mathbb{R}^n . Let also S be a closed surface, at each point y of which one places a point detector that measures the value of the pressure $p(y, t)$ at any moment $t > 0$. It is usually assumed (and this is crucial for the validity of the formulas of [10, 11, 24, 42]) that the object (and thus the support of $f(x)$) is surrounded by S . We will not need this assumption here.

We assume that the ultrasound speed $c(x)$ is known (e.g., being determined by transmission ultrasound measurements [17]). Then, the pressure wave $p(x, t)$ satisfies the following problem for the standard wave equation [7, 37, 41]:

$$\begin{cases} p_{tt} = c^2(x)\Delta_x p, t \geq 0, x \in \mathbb{R}^3 \\ p(x, 0) = f(x), \\ p_t(x, 0) = 0 \end{cases} \quad (1)$$

The task is to recover the initial value $f(x)$ at $t = 0$ of the solution $p(x, t)$ from the measured data, which we will call $g(y, t)$. Incorporating the data obtained from the measurement, the set of equations (1) extends to become

$$\begin{cases} p_{tt} = c^2(x)\Delta_x p, t \geq 0, x \in \mathbb{R}^3 \\ p(x, 0) = f(x), \\ p_t(x, 0) = 0 \\ p(y, t) = g(y, t), y \in S \times \mathbb{R}^+ \end{cases} \quad (2)$$

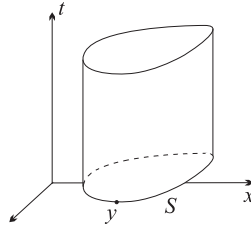


Figure 2: An illustration to (2).

The problem now is to find the initial value $f(x)$ in (2) from the knowledge of the lateral data $g(y, t)$ (see Figure 2). Although at a first glance one might think that we have insufficient data, this is in fact not true. Indeed, there is an additional restriction that the solution holds in the whole space, not just inside the cylinder $S \times \mathbb{R}^+$. In most cases, with this additional information, the data is sufficient for recovery of $f(x)$ (see [22] and references therein for the case of constant speed, and further in this paper for the case of a variable speed under a non-trapping condition). An additional sometimes useful comment (albeit we are not going to use it) is that p can be extended as an even function of time, after which it will satisfy the wave equation for all values of t . The same applies to the data g that can also be extended to an even function in t .

We would like to notice that in the case of constant speed, due to the well known Poisson-Kirchhoff formulas [6, Ch. VI, Section 13.2, Formula (15)] for the solution of (1), there is close relation (essentially, an equivalence) between this problem and inversion of the spherical mean operator with centers of spheres of integration located on S (see, e.g., [3, 22] and references in these papers). This is why many previous considerations dealt with the spherical mean operator rather than the wave equation, which allowed for the well developed techniques (or at least hints) of integral geometry to be applied. We cannot rely upon such techniques, since in our case the wave speed is not assumed to be constant.

The standard list of questions one would like to ask in any tomographic problem, includes uniqueness of reconstruction, inversion formulas and algorithms, their stability, incomplete data problems, etc. In this text, we will address only uniqueness of reconstruction and inversion. One can find a survey of known results on other issues and a bibliography in [22].

3 Discussion of the problem and the main results

In order not to obstacle consideration, we will assume for simplicity that the sound speed $c(x)$ and the initial function $f(x)$ to be recovered are infinite differentiable, an assumption that can be easily significantly weakened. The closed *observation surface* S of transducers' locations is assumed to be sufficiently "nice". E.g., assuming that it consists of finitely many transversally

intersecting smooth pieces (e.g., a cube), would suffice both for practical applications and for our analytic needs, albeit this condition can be weakened much further. Thus, we will not dwell on this issue. Notice, that unlike the case of all known backprojection type formulas in TAT, we do not assume that the function to be reconstructed is supported inside the observation surface S , on which the transducers are located.

The truly significant assumptions are the following:

1. Support of $f(x)$ is compact.
2. The sound speed is strictly positive $c(x) > c > 0$ and such that $c(x) - 1$ has compact support, i.e. $c(x) = 1$ for large x .
3. Consider the Hamiltonian system in $\mathbb{R}_{x,\xi}^{2n}$ with the Hamiltonian $H = \frac{c^2(x)}{2} |\xi|^2$:

$$\begin{cases} x'_t = \frac{\partial H}{\partial \xi} = c^2(x) \xi \\ \xi'_t = -\frac{\partial H}{\partial x} = -\frac{1}{2} \nabla (c^2(x)) |\xi|^2 \\ x|_{t=0} = x_0, \xi|_{t=0} = \xi_0. \end{cases} \quad (3)$$

The solutions of this system are called *bicharacteristics* and their projections into \mathbb{R}_x^n are *rays*.

We will assume that the **non-trapping condition** holds, i.e. that all rays (with $\xi_0 \neq 0$) tend to infinity when $t \rightarrow \infty$.

The first assumption is very important, albeit it could be replaced by a sufficiently fast decay. Without a sufficiently fast decay condition, there is no uniqueness of reconstruction [1, 3, 22]. The second condition assumes that the medium outside a bounded domain is homogeneous. The third one prohibits trapping. If trapping does occur, then it is naturally expected that solvability of standard inverse problems and problems like controllability should most probably fail.

Let us discuss first some peculiarities of the inverse problem of recovering the initial value $f(x)$ from the lateral boundary data $g(y, t)$ in (2). As we have mentioned already, if there were nothing special about the set-up of this problem (i.e., if we were dealing with boundary value problem for the wave equation in the cylinder), such an inverse problem would not be uniquely solvable. Indeed, the lateral data alone is clearly insufficient. The special situation is that we are dealing in fact with an *observability* problem. Namely,

the solution $p(x, t)$ satisfies the wave equation in the whole space, not just inside the surface S , albeit we observe the solution on S only. Since the initial function is compactly supported, and since appropriate non-trapping condition will be imposed, the energy will be locally decaying (e.g., [26, 38, 39]). Thus, the energy of the solution p inside S will be decaying. The rate of this decay is different in odd and even dimensions ([8, 38, 39]), exponential in the odd and power in the even case. In both cases, the solution is summable with respect to time as a function in appropriate functional spaces in the domain B bounded by S . We will see that this influences the solvability of the problem significantly.

3.1 A test case: constant sound speed in an odd dimension

We start with the simplest case when the dimension n is odd and the sound speed is constant and equal to 1. We also assume here that the initial function $f(x)$ is smooth and compactly supported. Then both the solution $p(x, t)$ and the boundary data $g(x, t)$ are smooth and, due to Huygens' principle, compactly supported in time, when $x \in B$. Here we denote by B the interior domain of S . Let now E be the operator of harmonic extension of functions from S to B , which, due to standard regularity theorems, is continuous from $H^s(S)$ to $H^{s+\frac{1}{2}}(B)$ for any $s > 0$. Then we can rewrite the problem (2) in terms of the function $u = p - Eg$:

$$\begin{cases} u_{tt} - \Delta_x u = -E(g_{tt}), t \geq 0, x \in B \\ u(x, 0) = f(x) - (Eg|_{t=0}), \\ u_t(x, 0) = 0 \\ u(y, t) = 0, y \in S \times \mathbb{R}^+ \end{cases} . \quad (4)$$

In particular, if the initial function is supported strictly inside B (which we do not need to assume), then $g|_{t=0} = 0$, and thus $u(x, 0) = f(x)$.

Let us now denote by Δ_D the Dirichlet Laplacian in B , defined in the standard way as an unbounded self-adjoint operator in $L^2(B)$.

Theorem 1. *The following representation of $u(x, t)$ holds:*

$$u(x, t) = \int_t^\infty (-\Delta_D)^{-\frac{1}{2}} \sin((t - \tau)(-\Delta_D)^{\frac{1}{2}}) E(g_{tt})(x, \tau) d\tau. \quad (5)$$

In particular,

$$f(x) = (Eg|_{t=0}) - \int_0^\infty (-\Delta_D)^{-\frac{1}{2}} \sin(\tau (-\Delta_D)^{\frac{1}{2}}) E(g_{tt})(x, \tau) d\tau. \quad (6)$$

Proof. First of all, the integral in (5) makes sense, since

$$(-\Delta_D)^{-\frac{1}{2}} \sin((t - \tau) (-\Delta_D)^{\frac{1}{2}})$$

is an uniformly bounded strongly continuous operator function in $L^2(B)$ and

$$\alpha(t) := E(g_{tt})(\cdot, t)$$

is a smooth compactly supported function of t with values in $L^2(B)$. Let $\{\lambda_k^2\}, \{\psi_k(x)\}$ be the spectrum and an orthonormal basis of eigenfunctions of $-\Delta_D$, where we assume $\lambda_k > 0$. We introduce notations

$$f_k := (f, \psi_k), u_k(t) := (u(\cdot, t), \psi_k), \text{ and } \alpha_k(t) := (\alpha(t), \psi_k)$$

for the Fourier coefficients of f , u , and α , with respect to the system $\{\psi_k\}$. Here (\cdot, \cdot) denotes the scalar product in $L^2(B)$.

It is sufficient to prove (5) coupled with any eigenfunction ψ_k :

$$u_k(t) = \int_t^\infty \left((-\Delta_D)^{-\frac{1}{2}} \sin((t - \tau) (-\Delta_D)^{\frac{1}{2}}) \alpha(\tau), \psi_k(x) \right) d\tau. \quad (7)$$

Using self-adjointness of $(-\Delta_D)^{-\frac{1}{2}} \sin((t - \tau) (-\Delta_D)^{\frac{1}{2}})$ and the eigenfunction property of ψ_k , this simplifies to

$$u_k(t) = \lambda_k^{-1} \int_t^\infty \sin((t - \tau) \lambda_k) \alpha_k(\tau) d\tau. \quad (8)$$

It is easy to check that the right hand side in (8) is the unique solution vanishing at infinity of the equation

$$u_{tt} = -\lambda_k^2 u - \alpha_k(t). \quad (9)$$

However, (9) coincides with the projection of (4) onto the eigenfunction ψ_k . Since the system of eigenfunctions is complete in $L^2(B)$, this proves the theorem. \square

Let us now try to remove the appearance of the extension operator E . In order to do so, we introduce the Fourier coefficients of the data:

$$g_k(t) = \langle g, \frac{\partial \psi_k}{\partial \nu} \rangle := \int_S g(x, t) \overline{\frac{\partial \psi_k}{\partial \nu}(x)} dx,$$

where ν is the external normal to S .

Lemma 2. *The following equality holds:*

$$(Eg)_k(t) = \lambda_k^{-2} g_k(t). \quad (10)$$

In particular,

$$\alpha_k(t) = \lambda_k^{-2} g_k''(t). \quad (11)$$

Proof. Indeed, $(Eg)_k = (Eg, \psi_k) = \lambda_k^{-2} (Eg, \Delta \psi_k)$. Using now Green's formula and harmonicity of Eg , one gets $(Eg)_k = \lambda_k^{-2} \langle g, \frac{\partial \psi_k}{\partial \nu} \rangle = \lambda_k^{-2} g_k(t)$. \square

The formulas (8) and (10)-(11) lead to

$$f_k = \lambda_k^{-2} g_k(0) - \lambda_k^{-3} \int_0^\infty \sin(\lambda_k t) g_k''(t) dt. \quad (12)$$

Now integration by parts in (12) proves the following

Theorem 3. *Function $f(x)$ in (2) can be reconstructed inside B from the data g in (2), as the following $L^2(B)$ -convergent series:*

$$f(x) = \sum_k f_k \psi_k(x), \quad (13)$$

where the Fourier coefficients f_k can be recovered using one of the following formulas:

$$\begin{cases} f_k = \lambda_k^{-2} g_k(0) - \lambda_k^{-3} \int_0^\infty \sin(\lambda_k t) g_k''(t) dt, \\ f_k = \lambda_k^{-2} g_k(0) + \lambda_k^{-2} \int_0^\infty \cos(\lambda_k t) g_k'(t) dt, \text{ or} \\ f_k = -\lambda_k^{-1} \int_0^\infty \sin(\lambda_k t) g_k(t) dt = -\lambda_k^{-1} \int_0^\infty \int_S \sin(\lambda_k t) g(x, t) \overline{\frac{\partial \psi_k}{\partial \nu}(x)} dx dt, \end{cases} \quad (14)$$

where

$$g_k(t) = \int_S g(x, t) \overline{\frac{\partial \psi_k}{\partial \nu}(x)} dx$$

and ν denotes the external normal to S .

Corollary 4. *The statement of the Theorem with the third of the coefficient formulas in (14) holds under much milder conditions on the function $f(x)$. For instance, it is sufficient to assume that $f \in H_{comp}^s(\mathbb{R}^n)$ for some $s > \frac{1}{2}$.*

Indeed, one can approximate such a function f in the space $H_{comp}^s(\mathbb{R}^n)$ by functions from $C_0^\infty(\mathbb{R}^n)$, thus getting convergence of the data $g(x, t)$ in $L_{comp}^2(S \times \mathbb{R})$. Then the third formula in (14) extends by continuity.

3.2 An abstract theorem

It is not hard to establish an abstract analog of Theorems 1 and 3.

Theorem 5. *Let H be a separable Hilbert space and A be a positive self-adjoint operator in H with discrete spectrum $\{\lambda_k^2\}$ and orthonormal basis of eigenvectors ψ_k . Suppose that function $u : \mathbb{R}^+ \rightarrow H$ solves the equation*

$$u_{tt} = -Au - \alpha(t) \tag{15}$$

and that the following conditions are satisfied:

1. *The function u and its derivative tend to zero when $t \rightarrow \infty$:*

$$\lim_{t \rightarrow \infty} \|u(t)\|_H = \lim_{t \rightarrow \infty} \|u_t(t)\|_H = 0.$$

2. *The function $\alpha(t)$ belongs to $L^1(\mathbb{R}^+, H)$.*

Then

1. *The following representation of $u(t)$ holds:*

$$u(t) = \int_t^\infty A^{-\frac{1}{2}} \sin((t - \tau)A^{\frac{1}{2}}) \alpha(\tau) d\tau. \tag{16}$$

In particular,

$$u(0) = - \int_0^\infty A^{-\frac{1}{2}} \sin(\tau A^{\frac{1}{2}}) \alpha(\tau) d\tau. \tag{17}$$

2. The coefficients u_k of the Fourier series expansion

$$u(0) = \sum_k u_k \psi_k$$

can be found as follows:

$$u_k = -\lambda_k^{-1} \int_0^\infty \sin(\tau \lambda_k) \alpha_k(\tau) d\tau,$$

where $\alpha_k(t) = (\alpha(t), \psi_k)_H$.

Proof. The proof repeats the proofs of Theorems 1 and 3. \square

3.3 An application to TAT in inhomogeneous media

Let us consider now the TAT problem (2) in the presence of an inhomogeneous background, assuming the conditions imposed on the sound speed $c(x)$ in the beginning of Section 3. Using the same harmonic extension trick as in the Section 3.1, we get

$$\begin{cases} u_{tt} - c^2(x) \Delta_x u = -E(g_{tt}), t \geq 0, x \in B \\ u(x, 0) = f(x) - (Eg|_{t=0}), \\ u_t(x, 0) = 0 \\ u(y, t) = 0, y \in S \times \mathbb{R}^+ \end{cases}. \quad (18)$$

Consider the Hilbert space $H = L^2(B, c^{-2}(x)dx)$, i.e., the weighted L^2 space with the weight $c^{-2}(x)$. On this space, the naturally defined operator

$$A = -c^2(x) \Delta$$

in B with zero Dirichlet conditions on S is self-adjoint, positive, and has discrete spectrum $\{\lambda_k^2\}(\lambda_k > 0)$ with eigenfunctions $\psi_k(x) \in H$. Now, since we are in fact dealing with the unobstacled whole space wave propagation (the surface S is not truly a boundary, but just an observation surface) and since we assumed the sound speed constant at infinity and non-trapping, the local energy decay type estimates of [38, 39] (see also [8, Theorem 2.104]) apply. They, in combination with the standard Sobolev trace theorems, imply in particular that the conditions of Theorem 5 both on $u(t) := u(\cdot, t) \in H$ and $\alpha(t) = E(g_{tt})$ are satisfied. We thus get the following

Theorem 6. 1. The function $f(x)$ in (18), and thus in (2) can be reconstructed inside B as follows:

$$f(x) = (Eg|_{t=0}) - \int_0^\infty A^{-\frac{1}{2}} \sin(\tau A^{\frac{1}{2}}) E(g_{tt})(x, \tau) d\tau. \quad (19)$$

2. Function $f(x)$ can be reconstructed inside B from the data g in (2), as the following $L^2(B)$ -convergent series:

$$f(x) = \sum_k f_k \psi_k(x), \quad (20)$$

where the Fourier coefficients f_k can be recovered using one of the following formulas:

$$\begin{cases} f_k = \lambda_k^{-2} g_k(0) - \lambda_k^{-3} \int_0^\infty \sin(\lambda_k t) g_k''(t) dt, \\ f_k = \lambda_k^{-2} g_k(0) + \lambda_k^{-2} \int_0^\infty \cos(\lambda_k t) g_k'(t) dt, \text{ or} \\ f_k = -\lambda_k^{-1} \int_0^\infty \sin(\lambda_k t) g_k(t) dt = -\lambda_k^{-1} \int_0^\infty \int_S \sin(\lambda_k t) g(x, t) \overline{\frac{\partial \psi_k}{\partial \nu}(x)} dx dt, \end{cases} \quad (21)$$

where

$$g_k(t) = \int_S g(x, t) \overline{\frac{\partial \psi_k}{\partial \nu}(x)} dx$$

and ν denotes the external normal to S .

Proof. The only thing that might have been different is deriving the Fourier coefficient representations (21) by coupling the expression $E(g_{tt})$ with ψ_k in (19) in the weighted L^2 space. This, however, causes no problem, since

$$\begin{aligned} & \int_B E(g_{tt})(x, t) \overline{\psi_k(x)} c^{-2}(x) dx \\ &= \lambda_k^{-2} \int_B E(g_{tt})(x, t) \overline{c^2(x) \Delta \psi_k(x)} c^{-2}(x) dx \\ &= \lambda_k^{-2} \int_B E(g_{tt})(x, t) \overline{\Delta \psi_k(x)} dx, \end{aligned}$$

from where the proof proceeds as before. □

Uniqueness of reconstruction of $f(x)$ inside B obviously follows from this Theorem. It, however, requires some additional arguments, if one wants to derive uniqueness of recovery of $f(x)$ outside S . We thus sketch an independent proof of uniqueness of reconstruction of the whole function $f(x)$, which does not rely upon the previous theorem.

Theorem 7. *Under the non-trapping conditions formulated above, compactly supported function $f(x)$ is uniquely determined by the data g . (No assumption of f being supported inside S is imposed.)*

Proof. Suppose that the data g vanishes. Then, inside B , due to local energy decay [8, 38, 39], $u(t)$, as an H -valued function, is integrable over \mathbb{R} . Taking Fourier transform in time, we get a continuous H -valued function $\hat{u}(\lambda)$ that satisfies the equation $(-\lambda^2 + A)\hat{u}(\lambda) = 0$ almost everywhere. However, the spectrum of A is discrete, and thus $\hat{u}(\lambda)$ must vanish for all but discrete set of values of λ . Due to continuity of $\hat{u}(\lambda)$, this implies that $\hat{u}(\lambda)$, and thus $u(t)$ vanish. This gives vanishing of f inside B . (This could also be derived immediately from the preceding theorem). Now, the function u extends to a solution of the wave equation outside B with matching Cauchy data on S . After Fourier transform in time, we get for each value of λ a solution $\hat{u}(x, \lambda)$ of the equation $c^2(x)\Delta\hat{u} + \lambda^2\hat{u} = 0$ in \mathbb{R}^n , vanishing inside S . Then standard uniqueness of continuation theorems (remember that c is smooth and non-vanishing) prove that \hat{u} , and hence u , is zero. \square

4 Further discussion and remarks

- The results of this paper are in some sense the realizations of the remarks made in [11] concerning inversion by solving that wave equation in reverse time direction and in [2] concerning inversions by eigenfunction expansions (improved and implemented in [25]).
- It is instructive to check how the last of the formulas (14) works in $1D$ in the case of constant speed. Let us assume that $B = [0, 1]$ and $f(x)$ is supported in B . Then $\lambda_k = k\pi$ and $\psi_k(x) = \sqrt{2}\sin k\pi x$. The d'Alembert formula claims that $u(x, t) = \frac{1}{2}(f(x-t) + f(x+t))$ and thus $g(0, t) = \frac{1}{2}f(t)$, $g(1, t) = \frac{1}{2}f(1-t)$ for $t \in [0, 1]$ and $g = 0$ for $t > 1$. The normal derivatives of the eigenfunctions are

$$\frac{\partial\psi_k}{\partial\nu}(0) = -\sqrt{2}k\pi, \frac{\partial\psi_k}{\partial\nu}(1) = \sqrt{2}(-1)^k k\pi.$$

Now substitution of these expression into the last of the formulas (14) after a short computation amounts to the standard formula for the Fourier coefficients of f with respect to the system ψ_k .

- Another instructive computation is to compare with the expansion formulas from [25] that were obtained under the assumption of a constant sound speed. Consider the 3D case. One needs to notice that the data used in [25], although denoted by $g(t)$ as ours, is different. In [25], it was, up to a factor 4π , the integral over the sphere of radius t , while ours corresponds to the solution of the wave equation (2). Taking this into account, our data (call it temporarily \tilde{g}), is equal to $\frac{d}{dt} \left(\frac{g(t)}{4\pi t} \right)$. Substitution of this \tilde{g} instead of g into the last formula of (14) and one integration by parts immediately lead to the formulas (10), (12), and (13) in [25]. Note however that, unlike in [25], no knowledge of the whole space Green's function is needed. In fact, our formulas hold in the case of variable non-trapping sound speed, when one cannot write this Green's function explicitly and where the method of [25] fails.
- Notice that, like in [25], and unlike [10, 11, 24, 42], the inversion procedures derived in this paper do not require the function $f(x)$ to be supported inside the observation surface S . Even if the (compact) support of f reaches outside S , reconstruction inside S is correct. It is known, however, that in this case the backprojection formulas from [10, 11, 24, 42] produce incorrect reconstructions inside S . The reason is that the formulas from [10, 11, 24, 42] have the information built in about the support of f being inside. In particular, they use, in our notations, only the values of t up to the diameter of the domain B . It is clear that in this case reconstruction of functions supported not necessarily inside S is not unique, since, due to the finite speed of propagation, information from all support of f might not reach S during this time. On the other hand, formulas of this paper, as well as of [25] use the information for all values of time t .
- It has been assumed throughout the paper that the initial velocity $p_t(x, 0)$ in (2) is equal to zero. It is clear, however, that this is of no importance for our considerations, and one can reconstruct the (possibly, non-zero) initial velocity as well, using formulas (5) and (16). These formulas were derived without using the assumption that $p_t(x, 0) = 0$.

- We went along with accepted models [7, 17, 37] of non-homogeneous media in TAT that lead to (2). In these models, variations of density are neglected. However, the abstract Hilbert space formalism that we provided, allows one to include also variations of density without much of a change in formulas and methods. This just leads to a more general operator A in $L^2(B)$.
- The integral formulas (5), (16), and (19) use functions of the Laplace (or a more general) operator in $L^2(B)$ with Dirichlet boundary conditions on S . It is clear how to use such formulas in combinations with eigenfunction expansions of the operator, as we did in this text. It is, however, less clear how such formulas could lead to analytic backprojection type inversions, such as the ones in [10, 11, 24, 42]. A Fourier transform consideration that leads to Helmholtz type equations suggests that there might be a link to [24], and thus one might hope to produce inversion formulas of [24] obtained for the case of S being a sphere, as a consequence of our formulas. The following simple comments might be useful for the further progress in this direction.

First of all, the gist of the consideration of Section 3.1 is the following simple relation (derived readily using Green's formula):

$$(\Delta_D u)_k = (\Delta u)_k + \int_S u \frac{\partial \psi_k}{\partial \nu} dA(x).$$

Secondly, reconstruction formulas (20)- (21) can be combined into the following integral formula:

$$f(x) = - \int_0^\infty \int_S \partial_{\nu_y} K(x, y, t) g(y, t) dA(y) dt, \quad (22)$$

where the kernel K is

$$K(x, y, t) = \sum_k \lambda_k^{-1} \sin(\lambda_k t) \psi_k(x) \psi_k(y). \quad (23)$$

It is readily checked that K satisfies the following initial-boundary value

problem:

$$\begin{cases} K_{tt} - c^2(y)\Delta_y u = 0, t \geq 0, y \in B \\ K(x, y, 0) = 0, \\ K_t(x, y, 0) = \delta(y - x), \\ K(x, y, t) = 0, y \in S \times \mathbb{R}^+ \end{cases} \quad (24)$$

In this context, the inversion formula (21) can be formally derived from the Green's formula:

$$\begin{aligned} 0 &= \int_{B \times [0, T]} [(K_{tt} - c^2(y)\Delta_y K)u - K(u_{tt} - c^2(y)\Delta u)]c^{-2}(y)dydt \\ &= \int_{S \times [0, T]} [K\partial_{\nu_y}g - \partial_{\nu_y}Kg]dA(y)dt + \int_{B \times (\{0\} \cup \{T\})} [K_t u - K u_t]dy, \end{aligned}$$

by substituting the initial-boundary conditions, letting $T \rightarrow \infty$ and using the vanishing of the solutions when $T \rightarrow \infty$.

It might be possible in the case of S being a sphere to convert the inversion formula for f into a backprojection type one.

- For the constant sound speed case, it has been known for quite some time [1, 3, 20] that any closed surface S provides uniqueness of reconstruction. The method of [3] is not applicable, since it relies upon constant sound speed. However, spectral methods of [1, 20] are applicable, as it is shown in the proof of Theorem 7.

Theorem 7 guarantees unique recovery of any compactly supported function $f(x)$, even if its support is not confined to B . However, microlocal arguments show that reconstruction should become unstable for some parts of f outside the observation surface S . This is related to the existence in this case of the so called “visible” or “audible” zones of the wave front set of $f(x)$, as well as those that are not “visible” (“audible”) [27, 34, 36]. See [2, 4, 22] for a brief discussion of this issue.

- Albeit this might not be clear from the text of this paper, what has led the authors to this work, was an approach through the range conditions of the spherical mean operator described in [2]. Indeed, the line of thought was that the whole possibility of inversion is based upon a very special type of the boundary data g involved. As we have already explained, using arbitrary functions as the data g would lead to impossibility of reconstruction. Thus, the basis of our approach was

to use our knowledge of the special features of the data g to derive inversions. It can be shown (albeit we do not do this in this text) that decay with time of solutions inside B is directly linked (in the case of constant sound speed) to the range descriptions of [2]. Continuing this consideration, one can also obtain some analogs of range descriptions for the case of variable sound speed.

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